Dynamical Systems
Tutorial 13: Symmetries and Reversors

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1 Symmetries

A flow is said to have a *symmetry* if there is a diffeomorphism, $S : M \to M$, that conjugates the flow to itself:

$$\varphi_t(S(z)) = S(\varphi_t(z)), \ \ t \in \mathbb{R}. \tag{6.24}$$

Since we assume that S is smooth, we can take the time derivative of this relation to obtain an equivalent requirement on the vector field associated with φ:

$$f(S(z)) = DS(z) f(z). \tag{6.25}$$

1.1 Continuous and discrete symmetries

$$\dot{r} = rh(r), \dot{\theta} = 1 \ (6.20)$$

Some symmetries, like a rotation symmetry, depend continuously upon a parameter and are thus called *continuous symmetries*. For example, the system (6.20) is obviously symmetric under the rotation

$$S_\psi(r, \theta) = (r, \theta + \psi) \tag{6.26}$$

for any angle ψ. For this case DS is the identity matrix, so (6.25) becomes $f(r, \theta + \psi) = f(r, \theta)$, which is satisfied for all ψ when f is a function of r only.

The collection of symmetries of a flow forms a group. This follows because the identity map is always a symmetry, and if S_1 and S_2 are symmetries of φ, then so is their composition $S_3 = S_1 \circ S_2$. Similarly, the inverse of a symmetry also satisfies (6.24) and therefore is also a symmetry. For example, the rotation symmetry (6.26) is a representation of the abstract rotation group, $O(2)$.

Discrete symmetries can also occur. For example, the system (6.11) is symmetric under the transformation $S(x, y) = (-x, -y)$, a rotation by π. To see this, note that for this case $DS = -I$, so (6.25) becomes $f(-x, -y) = -f(x, y)$, which is obviously satisfied by (6.11). The symmetry group in this case has two elements, the identity and S, and is called $\mathbb{Z}^2$. Much more about the implications of the existence of a nontrivial symmetry group can be found in (Field and Golubitsky 1995; Golubitsky and Stewart 2002).

$$\dot{x} = y^2x - x^2y, \dot{y} = x^3 + y^3 \ (6.11)$$

1.2 Reversors

Another type of symmetry that commonly occurs is a *time reversal* or *reversing* symmetry—when the motion backward in time is equivalent to that forward in time. Thus, a system is said to have reversing symmetry if there is a diffeomorphism, S (the reversor), that conjugates the flow to its inverse so that $\varphi_{-t}(S(z)) = S(\varphi_t(z))$. Again, this is equivalent to a requirement on the vector field

$$-f(S(z)) = DS(z)f(z). \tag{6.27}$$

This implies that in the new coordinate system, $\zeta = S(z)$, the differential equation $\dot{z} = f(z)$ becomes

$$\dot{\zeta} = DS(z)\dot{z} = DS(z)f(z) = -f(S(z)) = -f(\zeta),$$

which is the same differential equation going backward in time.

In many cases the reversor S is an involution, i.e., $S^2 = S \circ S = id$. For example, for mechanical Hamiltonian systems (recall §1.4) of the form

$$H(x, y) = \frac{1}{2}y^2 + V(x),$$

the involution $S(x, y) = (x, -y)$ reverses the momentum, y, and is equivalent to reversing time. Note also that in this case S is orientation reversing, $\det(DS) = -1 < 0$.

The fixed set of a reversor S is

$$\mathrm{Fix}(S) = \{z : z = S(z)\}.$$

An orbit that intersects $\mathrm{Fix}(S)$ is a *symmetric orbit*. In particular, a symmetric equilibrium is a point $z^* \in \mathrm{Fix}(S) \cap \{f(z) = 0\}$. Not every orbit is symmetric; however, every orbit has a symmetric partner (see Exercise 5).

It can be shown that the fixed set of any orientation-reversing involution in $\mathbb{R}^2$ is a curve, $C = \mathrm{Fix}(S)$ (MacKay). If this is the case, then whenever z^* is a symmetric, linear center, it must be a true center of the nonlinear system.

Lemma 1. *Suppose $\dot{z} = f(z)$ is reversible with reversor S and Fix(S) is a curve that contains an equilibrium z^* that is a linear center. Then z^* is a topological center.*

Recall, for a linear center at the origin:

▷ *Topological center*: there is a $\delta > 0$ such that every trajectory in $B_\delta(0)\setminus\{0\}$ is a closed loop enclosing the origin.

Proof idea: Close to the linear center, in polar coordinates, the angle θ must increase monotonically (see Meiss). Hence, for a point $z(0) \in \text{Fix}(S)$ in this neighborhood the orbit must return to $\text{Fix}(S)$ (roughly after an increase by π). Denote the time at which this first return happens τ. Then the reflection $\zeta(t) = S(z(t))$ of this orbit segment touches $\text{Fix}(S)$ at $z(0)$ and $z(\tau)$. However $\zeta(t)$ is a solution beginning at $z(0)$ and going backwards in time, and so the curve $\gamma = \{\varphi_t(z(0)) : -\tau \leq t \leq \tau\}$ is a closed loop and by uniqueness must be a periodic orbit with period 2τ.

Example: The system

$$\begin{aligned} \dot{x} &= -y + \alpha x^2 y, \\ \dot{y} &= x + \beta y^2 x^2 \end{aligned} \tag{6.28}$$

has the reversor $S(x, y) = (x, -y)$ since

$$DSf(x, y) = (-y+\alpha x^2 y, -x-\beta y^2 x^2) = -(-(-y)+\alpha x^2(-y), x+\beta(-y)^2 x^2) = -f(S(x, y)).$$

Note that the fixed curve for S is the x-axis, and since the origin is a symmetric fixed point, Lemma 6.4 implies it is a center. A phase portrait is shown in Figure 6.11. When $\alpha > 0$, this system also has a pair of saddle equilibria.

1.3 Meiss Ex 6.5

A flow φ has a reversor S and an orbit $\Gamma = \{\varphi_t(x) \mid t \in \mathbb{R}\}$.

(a) Show that $\bar{\Gamma} = \{S \circ \varphi_{-t}(x) \mid t \in \mathbb{R}\}$ is also an orbit of φ.

Since S is a reversor we know that:

$$\varphi_{-t}(S(z)) = S(\varphi_t(z)) \tag{1}$$

Hence:

$$S \circ \varphi_{-t}(x) = S(\varphi_{-t}(x)) = \varphi_{-(-t)}(S(x)) = \varphi_t(S(x)) \tag{2}$$

Denote $y = S(x)$, then $\bar{\Gamma}$ is of the form:

$$\bar{\Gamma} = \{\varphi_t(y) \mid t \in \mathbb{R}\} \tag{3}$$

i.e. $\bar{\Gamma}$ is also an orbit of φ.

(c) Suppose $\Gamma \cap \text{Fix}(S) \neq \emptyset$. Show that Γ and $\bar{\Gamma}$ coincide.

$$\text{Fix}(S) = \{z \mid S(z) = z\} \tag{4}$$

Consider $z^* \in \Gamma \cap \text{Fix}(S)$. As $z^* \in \Gamma$ we can write:

$$\Gamma = \{\varphi_t(z^*) \mid t \in \mathbb{R}\} \tag{5}$$

We saw in (a) that $\bar{\Gamma} = \{\varphi_t(S(x)) \mid t \in \mathbb{R}\}$. Since $\{\varphi_{-t}(x) \mid t \in \mathbb{R}\} = \{\varphi_{-t}(z^*) \mid t \in \mathbb{R}\}$ we can reach in the same way the result that:

$$\bar{\Gamma} = \{\varphi_t(S(z^*)) \mid t \in \mathbb{R}\} \tag{6}$$

Since $z^* \in \text{Fix}(S)$ we have:

$$\bar{\Gamma} = \{\varphi_t(z^*) \mid t \in \mathbb{R}\} = \Gamma \tag{7}$$

i.e. the orbits coincide.

1.4 Meiss Ex 6.6

(a) Show that if x^* is a symmetric equilibrium of a reversible system, then whenever λ is an eigenvalue of the linearization at x^*, so is $-\lambda$.

Denote the system $\dot{x} = f(x)$ and the reversor of the system as S. Since the system is reversible, we know:

$$-f(S(z)) = DS(z)f(z) \tag{8}$$

Differentiating:

$$-Df(S(z)) \cdot DS(z) = D^2S(z)f(z) + DS(z) \cdot Df(z) \tag{9}$$

The linearization at x^* is:

$$\dot{x} = Df(x^*)x \tag{10}$$

Since x^* is symmetric, we know $S(x^*) = x^*$. Substituting $z = x^* = S(z)$ in the previous equation, we have:

$$-Df(x^*) \cdot DS(x^*) = D^2S(x^*)f(x^*) + DS(x^*) \cdot Df(x^*) \tag{11}$$

Since x^* is an equilibrium, $f(x^*) = 0$. Since S, being a symmetry, is a diffeomorphism, we know that $DS(z)$ is invertible and hence there exists $(DS(x^*))^{-1}$. Multiplying the above equation from the left by $(DS(x^*))^{-1}$, we see that:

$$Df(x^*) = (DS(x^*))^{-1}(-Df(x^*))DS(x^*) \tag{12}$$

And so $Df(x^*)$ is similar to $-Df(x^*)$ and they have the same eigenvalues. However, the eigenvalues of $-Df(x^*)$ are simply minus the eigenvalues of $Df(x^*)$, and so if λ is an eigenvalue of $Df(x^*)$ then $-\lambda$ is an eigenvalue of $-Df(x^*)$, and hence of $Df(x^*)$ as well.

It is highly recommended that you solve the rest of exercises 6.5 and 6.6 at home for further practice on the subject of symmetries.

Bibliography

- Meiss, J. D. (2007). Differential dynamical systems, chapter 6